MIDTERM: INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: **6th March 2013**
The Total points is **115** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (15 points) When is a ring called a reduced ring? Define integral domain. Let R be a ring. Show that if R is an integral domain then it is reduced. Is the converse true?

(2) (10 points) Let I denote the defining ideal of an algebraic set X in the affine n-space $\mathbb{A}_k^n$ where k is an algebraically closed field. Recall that the cordinate ring of X is $R = k[X_1, \ldots, X_n]/I$. Show that R is a reduced ring.

(3) (20 points) Let B be a ring and A a subring of B. Let P be a prime ideal in A and $S = A \setminus P$. Let K be the field of fraction of A/P.
a) Show that $S^{-1}A/PS^{-1}A$ is isomorphic to K.
b) Show that $B \otimes_A K$ is isomorphic to $S^{-1}B/PS^{-1}B$.

(4) (20 points) Let R be a ring and $f : M \to N$ and $g : N \to P$ be R-module homomorphisms. When is the sequence $A \to^f B \to^g C$ called a complex? When is it called exact? Let F be a (finitely generated) free R-module and

$$0 \to A \to B \to C \to 0$$

be an exact sequence of R-modules. Show that:
a) $Hom_R(M \oplus N, A) \cong Hom_R(M, A) \oplus Hom_R(N, A)$ for any R-modules M and N.
b) $0 \to Hom_R(F, A) \to Hom_R(F, B) \to Hom_R(F, C) \to 0$ is exact.

(5) (20 points) Let A be an integral domain. When is A said to be a normal domain? Let $A \subset B$ be domains and $\alpha \in B$ be integral over A. Let K be the field of fractions of A and assume that $K(\alpha)/K$ is a separable extension. Show that the minimal polynomial of α over K have coefficients in A.

(6) (20 points) Let k be an algebraically closed field. Let $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ be affine algebraic varieties. Let $F : X \to Y$ be a map. When is F said to be a morphism of algebraic varieties. Show that there exist polynomials $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$ such that for all $\mathbf{a} = (a_1, \ldots, a_n) \in X$ the m-tuple $F(\mathbf{a}) = (f_1(\mathbf{a}), \ldots, f_m(\mathbf{a}))$.

(7) (10 points) Prove or disprove: Let $\phi : A \to B$ be a homomorphism of rings. Let $\mathbf{m}$ be maximal ideal of B then $\phi^{-1}(\mathbf{m})$ is a maximal ideal of A.